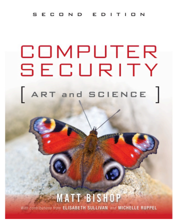
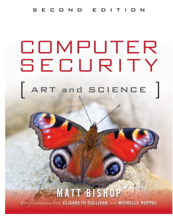


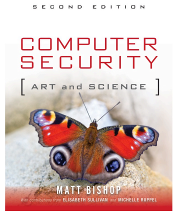
Lattices

Appendix A



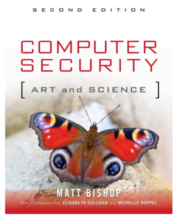
Outline

- Overview
- Definitions
- Lattices
- Examples



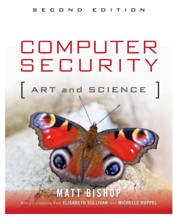
Overview

- Lattices used to analyze several models
  - Bell-LaPadula confidentiality model
  - Biba integrity model
- A lattice consists of a set and a relation
- Relation must partially order set
  - Relation orders some, but not all, elements of set



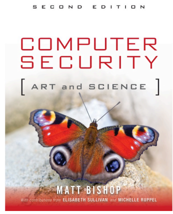
Sets and Relations

- S set, R: $S \times S$ relation
  - If $a, b \in S$, and $(a, b) \in R$, write aRb
- Example
  - $I = \{ 1, 2, 3 \}$; R is $\leq$
  - $R = \{ (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3) \}$
  - So we write $1 \leq 2$ and $3 \leq 3$ but not $3 \leq 2$



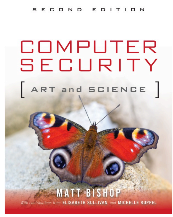
Relation Properties

- Reflexive
  - For all $a \in S$, aRa
  - On I, $\leq$ is reflexive as $1 \leq 1$, $2 \leq 2$, $3 \leq 3$
- Antisymmetric
  - For all $a, b \in S$, $aRb \wedge bRa \Rightarrow a = b$
  - On I, $\leq$ is antisymmetric as $1 \leq x$ and $x \leq 1$ means $x = 1$
- Transitive
  - For all $a, b, c \in S$, $aRb \wedge bRc \Rightarrow aRc$
  - On I, $\leq$ is transitive as $1 \leq 2$ and $2 \leq 3$ means $1 \leq 3$



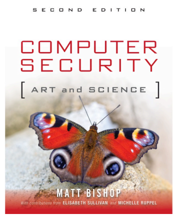
Example

- $\mathbb{C}$ set of complex numbers
- $a \in \mathbb{C} \Rightarrow a = a_{\mathbf{R}} + a_{\mathbf{I}}i$, where $a_{\mathbf{R}}$, $a_{\mathbf{I}}$ integers
- $a \leq_{\mathbf{C}} b$ if, and only if, $a_{\mathbf{R}} \leq b_{\mathbf{R}}$ and $a_{\mathbf{I}} \leq b_{\mathbf{I}}$
- $a \leq_{\mathbf{C}} b$ is reflexive, antisymmetric, transitive
  - As ≤ is over integers, and $a_{\mathbf{R}}$, $a_{\mathbf{I}}$ are integers



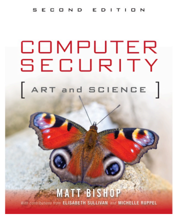
Partial Ordering

- Relation *R* orders some members of set *S*
  - If all ordered, it's a total ordering
- Example
  - $\leq$ on integers is total ordering
  - $\leq_{\mathbb{C}}$ is partial ordering on $\mathbb{C}$
    - Neither $3+5i \leq_{\mathbb{C}} 4+2i$ nor $4+2i \leq_{\mathbb{C}} 3+5i$ holds



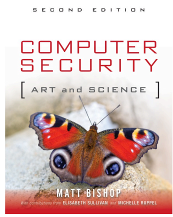
Upper Bounds

- For *a*, *b* ∈ *S*, if *u* in *S* with *aRu*, *bRu* exists, then *u* is an *upper bound*
  - A *least upper bound* if there is no *t* ∈ *S* such that *aRt*, *bRt*, and *tRu*
- Example
  - For 1 + 5*i*, 2 + 4*i* ∈ $\mathbb{C}$
    - Some upper bounds are 2 + 5*i*, 3 + 8*i*, and 9 + 100*i*
    - Least upper bound is 2 + 5*i*



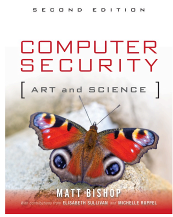
Lower Bounds

- For *a*, *b* $\in$ *S*, if *l* in *S* with *lRa*, *lRb* exists, then *l* is a *lower bound*
    - A *greatest lower bound* if there is no *t* $\in$ *S* such that *tRa*, *tRb*, and *lRt*
- Example
    - For 1 + 5*i*, 2 + 4*i* $\in \mathbb{C}$
        - Some lower bounds are 0, –1 + 2*i*, 1 + 1*i*, and 1+4*i*
        - Greatest lower bound is 1 + 4*i*



Lattices

- Set *S*, relation *R*
  - *R* is reflexive, antisymmetric, transitive on elements of *S*
  - For every $s, t \in S$, there exists a greatest lower bound under *R*
  - For every $s, t \in S$, there exists a least upper bound under *R*

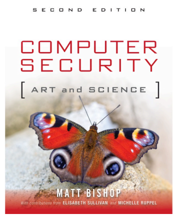


Example

- $S = \{ 0, 1, 2 \}$; $R = \leq$ is a lattice
  - R is clearly reflexive, antisymmetric, transitive on elements of S
  - Least upper bound of any two elements of S is the greater of the elements
  - Greatest lower bound of any two elements of S is the lesser of the elements

Picture

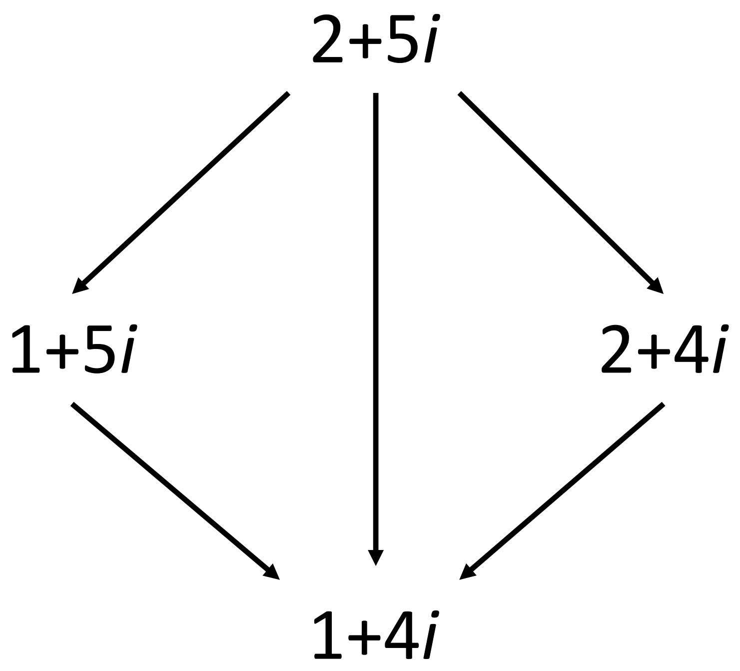
Arrows represent ≤; this forms a total ordering



Example

- $\mathbb{C}$, $\leq_{\mathbb{C}}$ form a lattice
  - $\leq_{\mathbb{C}}$ is reflexive, antisymmetric, and transitive
    - Shown earlier
  - Least upper bound for *a* and *b*:
    - $c_R = \max(a_R, b_R)$, $c_I = \max(a_I, b_I)$; then $c = c_R + c_I i$
  - Greatest lower bound for *a* and *b*:
    - $c_R = \min(a_R, b_R)$, $c_I = \min(a_I, b_I)$; then $c = c_R + c_I i$

Picture



Arrows represent $\leq_{\mathbb{C}}$