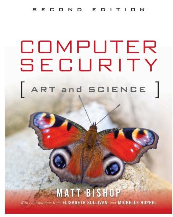
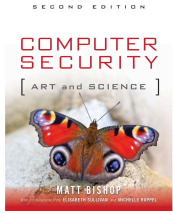


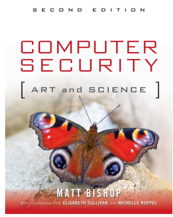
Euclidean Algorithm

Appendix B



Outline

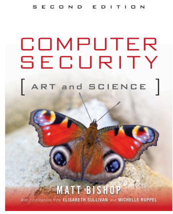
- Overview
- Definitions
- Lattices
- Examples



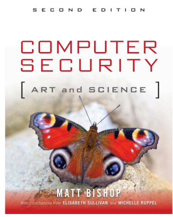
Overview

- Solving modular equations arises in cryptography
- Euclidean Algorithm
- From Euclid to solving $ax \bmod n = 1$
- From $ax \bmod n = 1$ to solving $ax \bmod n = b$

Euclidean Algorithm



- Given positive integers *a* and *b*, find their greatest common divisor
- Idea
  - if *x* is the greatest common divisor of *a* and *b*, then *x* divides $r = a - b$
  - reduces problem to finding largest *x* that divides *r* and *b*
  - iterate

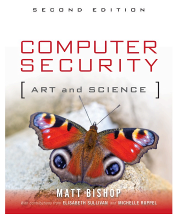


Relation Properties

- Take $a = 15$, $b = 12$

a	b	q	r		
15	12	1	3	$q = 15/12 = 1$	$r = 15 - 1\times12 = 3$
12	3	4	0	$q = 12/3 = 4$	$r = 12 - 4\times3 = 0$

- So $gcd(15, 12) = 3$
  - The b for which r is 0



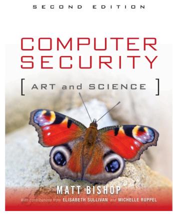
Example 2

- Take a = 35731, b = 25689

a	b	q	r		
35731	24689	1	11042	q = 35731/24689 = 1	r = 35731–1×24689 = 11042
24689	11042	2	2605	q = 24689/11042 = 2	r = 24689–2×11042 = 2605
11042	2605	4	622	q = 11042/2605 = 4	r = 11042–4×2605 = 622
2605	622	4	117	q = 2605/622 = 4	r = 2605–4×622 = 117
622	117	5	37	q = 622/117 = 5	r = 622–5×117 = 37
117	37	3	6	q = 117/37 = 3	r = 117–3×37
37	6	6	1	q = 37/6 = 6	r = 37–6×6 = 1
6	1	6	0	q = 6/1 = 6	r = 6–6×1 = 0

Pseudocode

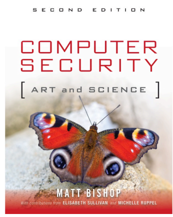
```
/* find gcd of a and b */
rprev := r := 1;
while r <> 0 do begin
  rprev := r;
  r := a mod b;
  write 'a = ', a, 'b =', b, 'q = ‘, a div b,
                                  'r = ', r, endline;
  a := b;
  b := r;
end;
gcd := rprev;
```



Extended Euclidean Algorithm

- Find two integers x and y such that

$$xa + yb = 1$$

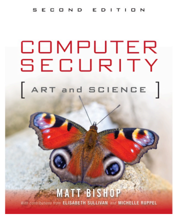


Example 1

- Find x and y such that $51x + 100y = 1$

u	x	y	q			
100	0	1				
51	1	0	100/51 = 1	$u = 100–1\times51 = 49$	$x = 0–1\times1 = –1$	$y = 1–1\times0 = 1$
49	–1	1	51/49 = 1	$u = 51–1\times49 = 2$	$x = 0–1\times1 = –1$	$y = 1–1\times0 = 1$
2	2	–1	49/2 = 24	$u = 49–24\times2 = 1$	$x = 1–1\times(–1)=2$	$y = 0–1\times1=–1$
1	–49	25	2/1 = 2	$u = 2–2\times1 = 0$	$x = –1–24\times1 = –49$	$y = 1–24\times(–1)=25$
0	100	–51			$x = 2–49\times2 = 100$	$y = –1–25\times2 = –51$

- So, $51 \times (–49) + 100 \times 25 = 1$
  - This is $–2499 + 2500 = 1$

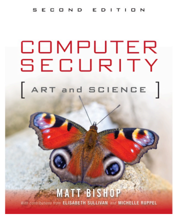


Example 2

- Find x and y such that $24689x + 35731y = 1$

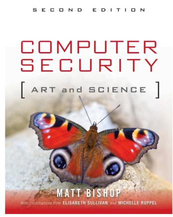
u	x	y	q			
35731	0	1				
24689	1	0	35731/24689 = 1	u = 35721–1×24689	x = 0–1×1	y = 1–1×
11042	–1	1	24689/11042 = 2	u = 24689–2×11042	x = 1–2×(–1)	y = 0–2×1
2605	3	–2	11042/2605 = 4	u = 11042–4×2605	x = –1–4×3	y = 1–4×(–2)
622	–13	9	2605/622 = 4	u = 2605–4×622	x = 3–4×(–13)	y = –2–4×9
117	55	–38	622/117 = 5	u = 622–5×117	x = –13–5×55	y = 9–5×(–38)
37	–288	199	117/37 = 3	u = 117–3×37	x = 55–3×(–288)	y = –38–3×199
6	919	–635	37/6 = 6	u = 37–6×6	x = –288–6×919	y = 199–6×(–635)
1	–5802	4009	6/1=6	u = 6–6×1	x = 919–6×(–5802)	y = –635–6×(4009)
0	35731	–24689				

So, 24689 × (–5802) + 35731 × 4009 = 1



Pseudocode

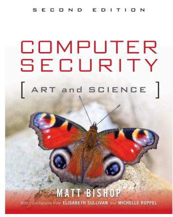
```
/* find x and y such that ax + by = 1, for given a and b */
uprev := a; u := b;
xprev := 0; x := 1; yprev := 1; y := 0;
write 'u = ', uprev, ' x = ', xprev, ' y = ', yprev,  endline;
write 'u = ', u, ' x = ', x, ' y = ', y;
while u <> 0 do begin
     q := uprev div u;
     write 'q = ', q, endline;
     utmp := uprev – u * q; uprev := u; u := utmp;
     xtmp := xprev – x * q; xprev := x; x := xtmp;
     ytmp := yprev – y * q; yprev := y; y := ytmp;
     write 'u = ', u, ' x = ', x, ' y = ', y;
end;
write endline;
x := xprev; y := yprev;
```



Solving ax mod $n = 1$

- If ax mod $n = 1$ then choose k such that $ax = 1 + kn$, or $ax - kn = 1$. If $b = -k$, then $ax + bn = 1$.
- Use extended Euclidean algorithm to solve for a

Example



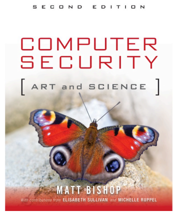
- Solve for x: $51x \bmod 100 = 1$
  - Recall (from earlier example)

    $51 \times (-49) + 100 \times 25 = 1$

    Then $x = -49 \bmod 100 = 51$
- Solve for x: $24689 \bmod 35731 = 1$
  - Recall (from earlier example)

    $24689 \times (-5802) + 35731 \times 4009 = 1$

    Then $x = -5802 \bmod 35731 = 29929$



Solving $ax \bmod n = b$

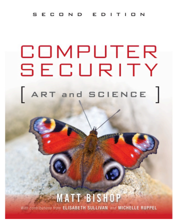
- A fundamental law of modular arithmetic:

$$xy \bmod n = (x \bmod n)(y \bmod n) \bmod n$$

so if x solves $ax \bmod n = 1$, then as

$$b(ax \bmod n) = a(bx) \bmod n = b$$

bx solves $ax \bmod n = b$



Example

- Solve for x: 51x mod 100 = 10
  - Recall (from earlier example) that if
    51y mod 100 = 1, then y = 51.
    Then x = 10 $\times$ 51 mod 100 = 510 mod 100 = 10
- Solve for x: 24689 mod 35731 = 1753
  - Recall (from earlier example) that if
    24689y mod 35731 = 1, then y = 29929.
    Then x = 1753 $\times$ 29929 mod 35731 = 12429